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# Explicit bounds for second-order difference equations and a solution to a question of Stević

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**Abstract**

This note gives explicit, applicable bounds for solutions of a wide class of second-order difference equations with nonconstant coefficients. Among the applications is an affirmative answer to a recent question of Stević.


*Keywords:* Explicit bounds; Applicable bounds; Second-order linear difference equations; Growth rates; Nonconstant coefficients

## 1. Introduction

This paper studies explicit, applicable growth rates for second-order difference equations. In particular, we will consider equations of the form

$$b_i = \big(2 + g(i-1)\big)b_{i-1} - \big(1 + h(i-1)\big)b_{i-2}, \tag{1}$$

for $i \geqslant 2$, and provide sharp inequalities for $\{b_i\}$ in terms of the sequences $\{g(i)\}$ and $\{h(i)\}$, and the initial values $b_0$ and $b_1$. Solutions of difference equations of the form in (1)

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have been studied by many authors (cf. [2–4,6–13]). Often the study has focused on the understanding of oscillatory or asymptotic behavior.

Our main theorem (Theorem 2) implies the following result which partially answers a question of Stević [8].

**Theorem 1.** *Suppose that* $c \geqslant 0$ *and* $\{b_i\}$ *satisfies* (1), *with* $g(i) = c/i^2$. *Then, for* $n \geqslant 0$,

$$|b_n| \leqslant \left(|b_0| + \frac{|b_1|}{\chi_c}\right) n^{k_c}, \tag{2}$$

*where*

$$k_c = \frac{1 + \sqrt{4c+1}}{2} \tag{3}$$

*and*

$$\chi_c = \begin{cases} 1, & \text{if } 0 \leqslant c \leqslant 2 \text{ or } c \geqslant 6, \\ \frac{2^{k_c} - 1 - \frac{1}{6}c(k_c - 2)(2^{k_c - 3} + 1)}{1 + c}, & \text{if } 2 < c < 6. \end{cases} \tag{4}$$

Note that $c = k_c(k_c - 1)$. Hence, by Lemma 2(a), in Section 3, for $2 \leqslant c \leqslant 6$ (i.e., $2 \leqslant k_c \leqslant 3$),

$$\chi_c \leqslant \frac{2^{k_c} - 1}{1 + k_c(k_c - 1)} \leqslant 1, \tag{5}$$

and by Lemma 2(c),

$$\chi_c \geqslant \frac{3}{1 + k_c(k_c - 1)} \geqslant \frac{3}{7}. \tag{6}$$

In [8], Stević proved that $b_n = O(n^{c+1})$ and correctly conjectured that $b_n = O(n^{k_c})$.

The main theorem here is the following.

**Theorem 2.** *Suppose* $B$ *and* $g$ *are positive functions satisfying the second-order differential equation*

$$B''(x) = g(x)B(x), \tag{7}$$

$B'''(x)$ *exists on* $x \geqslant 1$, $\{b_i\}$ *is a solution to the difference equation in* (1), *with* $h(1) > -1$, *and*

$$h(n) \leqslant g(n), \tag{8}$$

*for all* $n \geqslant 2$. *Let* $V(2) = 0$ *and*

$$V(n) = \frac{1}{6}\left(\sum_{i=2}^{n-1} \min_{i-1<\eta_i<i<\zeta_i<i+1} \left\{B'''(\zeta_i) - B'''(\eta_i)\right\}\right) + H(n), \tag{9}$$

*for* $n \geqslant 3$, *where*

$$H(n) = \sum_{\substack{2 \leqslant i \leqslant n-1 \\ h(i)<0}} h(i)B(i-1). \tag{10}$$

*In addition, suppose there exist positive constants, $c_0$ and $c_1$, satisfying*

$$0 < c_0 \leqslant \min\left\{\frac{B(2)-B(1)+V(n)}{1+h(1)}, B(1)-B(0)\right\}, \tag{11}$$

$$0 < c_1 \leqslant \min\left\{\frac{B(2)-B(1)+V(n)}{1+g(1)}, B(1)-B(0)\right\}, \tag{12}$$

*for all $n \geqslant 3$, then*

$$|b_n| \leqslant \left(\frac{|b_0|}{c_0} + \frac{|b_1|}{c_1}\right)B(n), \tag{13}$$

*for $n \geqslant 0$.*

Note that from the conditions in Theorem 2, it follows that $B$ is increasing.

If $B'''(x)$ is nondecreasing in $x$ and $h \equiv 0$, then $V(n) \geqslant 0$, for all $n$, and we obtain the following corollary.

**Corollary 1.** *Suppose $B$ and $g$ are positive functions satisfying* (7)*, $B'''(x)$ is nondecreasing on $x \geqslant 1$, and $\{b_i\}$ is a solution to the difference equation in* (1) *with $h \equiv 0$. In addition, suppose that there exist positive constants, $c_0$ and $c_1$, satisfying*

$$0 < c_0 \leqslant \min\{B(2)-B(1), B(1)-B(0)\}, \tag{14}$$

$$0 < c_1 \leqslant \min\left\{\frac{B(2)-B(1)}{1+g(1)}, B(1)-B(0)\right\}, \tag{15}$$

*then* (13) *is satisfied for all $n \geqslant 0$.*

Similarly, if $B'''(x)$ is nonincreasing in $x$ and $h \equiv 0$, we have

$$\begin{aligned} V(n) &= \frac{1}{6}\left(\sum_{i=2}^{n-1} \min_{i-1<\eta_i<i<\zeta_i<i+1} \{B'''(\zeta_i) - B'''(\eta_i)\}\right) \\ &= \frac{1}{6}\left(\sum_{i=2}^{n-1} \big(B'''(i+1) - B'''(i-1)\big)\right) \\ &= \frac{1}{6}\big(B'''(n) + B'''(n-1) - B'''(2) - B'''(1)\big), \end{aligned} \tag{16}$$

for $n \geqslant 3$, and Theorem 2 leads to the following corollary.

**Corollary 2.** *Suppose $B$ and $g$ are positive functions satisfying* (7)*, $B'''(x)$ is nonincreasing on $x \geqslant 1$, with*

$$\inf_{n \geqslant 1}\left\{\frac{1}{6}\big(B'''(n) + B'''(n-1) - B'''(2) - B'''(1)\big)\right\} \geqslant C, \tag{17}$$

Table 1
Constants $c_0$ and $c_1$ for some pairs $(g, B)$

| $B(x)$ | $g(x)$ | $B'''(x)$ | $c_0$ | $c_1$ |
|---|---|---|---|---|
| $x^5$ | $20/x^2$ | $\uparrow$ | 1 | 1 |
| $\sqrt{x}I_1(2\sqrt{x})$ | $1/x$ | $\uparrow$ | 1.5906 | 1.5906 |
| $x\ln(x+1)$ | $(x+2)/((x+1)^2\ln(x+1))$ | $\uparrow$ | 0.693147 | 0.693147 |
| $xe^x$ | $1+2/x$ | $\uparrow$ | 2.71828 | 2.71828 |
| $x^{5/2}$ | $3.75/x^2$ | $\downarrow$ | 1 | 0.86808 |

*for some $C$, and $\{b_i\}$ is a solution to the difference equation in* (1), *with $h \equiv 0$. In addition, suppose that there exist positive constants, $c_0$ and $c_1$, satisfying*

$$\begin{aligned} &0 < c_0 \leqslant \min\bigl\{B(2) - B(1) + C, B(1) - B(0)\bigr\}, \\ &0 < c_1 \leqslant \min\left\{\frac{B(2) - B(1) + C}{1 + g(1)}, B(1) - B(0)\right\}, \end{aligned} \tag{18}$$

*then* (13) *is satisfied for all $n \geqslant 0$.*

Note that if $B'''(x) \geqslant 0$, for all $x$, then we may take $C = -\frac{1}{6}(B'''(2) + B'''(1))$ in (17).

**Example.** In [8], Stević also proved that if $g(i) = 1/i$ for $i \geqslant 1$, then $b_n = O(ne^n)$. As noted in Table 1, for that particular $g$, we actually have

$$b_n = O\bigl(\sqrt{n}I_1\bigl(2\sqrt{n}\bigr)\bigr), \tag{19}$$

where $I_k(z)$ denotes the modified Bessel function of the first kind (cf. [1]). To see how the two bounds compare, note that

$$\lim_{n\to\infty}\bigl(\sqrt{n}I_1\bigl(2\sqrt{n}\bigr)\bigr)^{1/n} = 1. \tag{20}$$

As shown in Table 1, a more appropriate $g$ for the bound $ne^n$ is given by $g(i) = 1 + 2/i$.

Table 1 gives several noteworthy examples of pairs $(g, B)$ with associated constants $c_0$ and $c_1$.

The remainder of the paper proceeds as follows. Section 2 comprises a proof of Theorem 2, while Section 3 includes a proof of Theorem 1 which uses Corollaries 1 and 2.

## 2. Proof of the main result

In this section we prove Theorem 2.

We will employ the following elementary lemma (cf. Mitrinović [5, p. 362]) which follows directly from Taylor's theorem.

**Lemma 1.** *Suppose $f$ is defined over the interval $(n-1, n+1)$. If $f'''(x)$ exists for $n-1 \leqslant x \leqslant n+1$, then*

$$f(n+1) - 2f(n) + f(n-1) = f''(n) + \frac{1}{6}\bigl(f'''(\zeta) - f'''(\eta)\bigr), \tag{21}$$

*for some*

$$n - 1 < \eta < n < \zeta < n + 1. \tag{22}$$

**Proof of Theorem 2.** Suppose $B$ and $g$ satisfy the assumptions of the theorem and define

$$\varepsilon_i = B(i) - b_i \quad (i \geqslant 0), \tag{23}$$

and

$$\Delta\varepsilon_{i-1} = \varepsilon_i - \varepsilon_{i-1} \quad (i \geqslant 1). \tag{24}$$

First, set $b_0 = -c_0$ and $b_1 = 0$. We will show that in this case, $\{b_i\}_{i>0}$ and $\{b_{i+1} - b_i\}_{i>0}$ are nonnegative sequences, i.e.,

$$b_{i+1} \geqslant b_i \geqslant 0, \tag{25}$$

for $i \geqslant 1$. Note that $b_1 = 0$, and since $h(1) > -1$, we have that $b_2 = (1 + h(1))c_0 > 0$. Thus assume (25) holds for $1 \leqslant i \leqslant M - 2$, for some $M \geqslant 3$. By (1), the induction hypothesis, and (8), we have

$$\begin{aligned} b_M - b_{M-1} &= \bigl(1 + g(M-1)\bigr)b_{M-1} - \bigl(1 + h(M-1)\bigr)b_{M-2} \\ &\geqslant g(M-1)b_{M-1} - h(M-1)b_{M-2} \\ &\geqslant g(M-1)b_{M-2} - h(M-1)b_{M-2} \\ &= \bigl(g(M-1) - h(M-1)\bigr)b_{M-2} \geqslant 0. \end{aligned} \tag{26}$$

Thus (25) holds for all $i \geqslant 1$.

Now, by (23) and (11), we have

$$\Delta\varepsilon_0 = \varepsilon_1 - \varepsilon_0 = B(1) - b_1 - \bigl(B(0) - b_0\bigr) = B(1) - B(0) - c_0 \geqslant 0. \tag{27}$$

Also, employing (1) and (11),

$$\begin{aligned} \Delta\varepsilon_1 &= B(2) - b_2 - \bigl(B(1) - b_1\bigr) \\ &= B(2) - B(1) - \bigl(\bigl(1 + g(1)\bigr)b_1 - \bigl(1 + h(1)\bigr)b_0\bigr) \\ &= B(2) - B(1) - \bigl(1 + h(1)\bigr)c_0 \geqslant 0. \end{aligned} \tag{28}$$

For $\Delta\varepsilon_2$, we have, for some $\zeta_2$ and $\eta_2$ satisfying $1 < \eta_2 < 2 < \zeta_2 < 3$,

$$\begin{aligned} \Delta\varepsilon_2 &= B(3) - B(2) - \bigl(\bigl(1 + g(2)\bigr)b_2 - \bigl(1 + h(2)\bigr)b_1\bigr) \\ &= \bigl(B(3) - 2B(2) + B(1)\bigr) - g(2)B(2) + g(2)\bigl(B(2) - b_2\bigr) \\ &\quad + \bigl(B(2) - b_2\bigr) - \bigl(B(1) - b_1\bigr) + h(2)b_1 \\ &= \frac{1}{6}\bigl(B'''(\zeta_2) - B'''(\eta_2)\bigr) + B''(2) - g(2)B(2) + g(2)\varepsilon_2 + \Delta\varepsilon_1 + h(2)b_1 \\ &= \frac{1}{6}\bigl(B'''(\zeta_2) - B'''(\eta_2)\bigr) + g(2)\varepsilon_2 + \Delta\varepsilon_1 + h(2)b_1, \end{aligned} \tag{29}$$

where we have used (1), Lemma 1, and (7). Repeating the process in (29), successively, to rewrite $\Delta\varepsilon_3, \Delta\varepsilon_4, \ldots, \Delta\varepsilon_{n-1}$, we obtain

$$\begin{aligned}
\Delta\varepsilon_{n-1} &= \varepsilon_n - \varepsilon_{n-1} \\
&= \frac{1}{6}\sum_{i=2}^{n-1}\bigl(B'''(\zeta_i) - B'''(\eta_i)\bigr) + \sum_{i=2}^{n-1} g(i)\varepsilon_i + \Delta\varepsilon_1 + \sum_{i=2}^{n-1} h(i)b_{i-1} \\
&\geqslant \frac{1}{6}\sum_{i=2}^{n-1}\bigl(B'''(\zeta_i) - B'''(\eta_i)\bigr) + \sum_{i=2}^{n-1} g(i)\varepsilon_i + \Delta\varepsilon_1 \\
&\quad + \sum_{\substack{2\leqslant i\leqslant n-1\\ h(i)<0}} h(i)b_{i-1} \quad \bigl(\text{by } (25)\bigr) \\
&= V(n) + \sum_{i=2}^{n-1} g(i)\varepsilon_i + \Delta\varepsilon_1 - \sum_{\substack{2\leqslant i\leqslant n-1\\ h(i)<0}} h(i)\varepsilon_{i-1} \\
&= V(n) + B(2) - B(1) - \bigl(1 + h(1)\bigr)c_0 + \sum_{i=2}^{n-1} g(i)\varepsilon_i \\
&\quad + \sum_{\substack{2\leqslant i\leqslant n-1\\ h(i)<0}} \bigl|h(i)\bigr|\varepsilon_{i-1} \geqslant \sum_{i=2}^{n-1} g(i)\varepsilon_i + \sum_{\substack{2\leqslant i\leqslant n-1\\ h(i)<0}} \bigl|h(i)\bigr|\varepsilon_{i-1},
\end{aligned} \tag{30}$$

where $\zeta_i$ and $\eta_i$ satisfy $i-1 < \eta_i < i < \zeta_i < i+1$, for $i \in \{2, 3, \ldots, n-1\}$. The inequality in (30) follows from (11).

Now, $\varepsilon_0 = B(0) - b_0 = B(0) + c_0 > 0$, and hence from (27) and (28), we obtain

$$\varepsilon_2 \geqslant \varepsilon_1 \geqslant \varepsilon_0 > 0. \tag{31}$$

Thus, assume $\varepsilon_i \geqslant 0$, for $0 \leqslant i \leqslant N-1$, for some $N \geqslant 3$. Then, by (30), the induction hypothesis, and the fact that $g$ is positive, we have

$$\varepsilon_N = \Delta\varepsilon_{N-1} + \varepsilon_{N-1} \geqslant \sum_{i=2}^{N-1} g(i)\varepsilon_i + \sum_{\substack{2\leqslant i\leqslant n-1\\ h(i)<0}} \bigl|h(i)\bigr|\varepsilon_{i-1} \geqslant 0, \tag{32}$$

and the induction is complete. Combining this with (25) gives

$$|b_n| \leqslant B(n), \tag{33}$$

for all $n \geqslant 0$.

A similar argument also holds when, in place of the sequence $\{b_i\}$, we consider the solution $\{b_i^*\}$ of (1) with starting values

$$b_0^* = 0 \quad \text{and} \quad b_1^* = c_1. \tag{34}$$

We then have

$$\left|b_n^*\right| \leqslant B(n), \tag{35}$$

for all $n \geqslant 1$.

To complete the proof, note that for the solution $\{b_i^\dagger\}$, with arbitrary starting values $b_0^\dagger$ and $b_1^\dagger$, we have

$$\begin{aligned}\left|b_i^\dagger\right| &= \left|\frac{b_0^\dagger}{-c_0}b_i + \frac{b_1^\dagger}{c_1}b_i^*\right| \leqslant \left|\frac{b_0^\dagger}{c_0}b_i\right| + \left|\frac{b_1^\dagger}{c_1}b_i^*\right| \leqslant \left|\frac{b_0^\dagger}{c_0}\right|B(i) + \left|\frac{b_1^\dagger}{c_1}\right|B(i) \\ &= \left(\left|\frac{b_0^\dagger}{c_0}\right| + \left|\frac{b_1^\dagger}{c_1}\right|\right)B(i). \end{aligned} \tag{36}$$

This completes the proof of Theorem 2. $\square$

## 3. A question of Stević

In this section, we employ Corollaries 1 and 2 to prove Theorem 1. First we require the following technical lemma.

**Lemma 2.** *Define, for $x \geqslant 1$, the functions $z$, $r_1$, and $r_2$ via*

$$z(x) \stackrel{\text{def}}{=} \frac{1}{6}x(x-1)(x-2), \qquad r_1(x) \stackrel{\text{def}}{=} 2^x - 2 - x(x-1), \quad \textit{and}$$
$$r_2(x) \stackrel{\text{def}}{=} 2^x - 4 - z(x)\bigl(2^{x-3}+1\bigr) = 2^{x-3}\bigl(8 - z(x)\bigr) - \bigl(4 + z(x)\bigr).$$

*We then have the following inequalities*:

(a) $r_1(x) \leqslant 0$ *for* $2 \leqslant x \leqslant 3$;
(b) $r_1(x) \geqslant 0$ *for* $1 \leqslant x \leqslant 2$, $x \geqslant 3$;
(c) $r_2(x) \geqslant 0$ *for* $2 \leqslant x \leqslant 3$.

**Proof.** Note that

$$r_1''(x) = 2^x \ln^2 2 - 2, \tag{37}$$

and hence $r_1''(x) \leqslant 0$ for $1 \leqslant x \leqslant x_0$ and $r_1''(x) \geqslant 0$ for $x \geqslant x_0$, where

$$x_0 \stackrel{\text{def}}{=} \frac{\ln\left(\frac{2}{\ln^2 2}\right)}{\ln 2}. \tag{38}$$

Since $1/4 < \ln^2 2 < 1/2$, the definition in (38) leads to $2 < x_0 < 3$ (in fact, $x_0 \approx 2.057532746$). The inequalities in (a) and (b) now follow via concavity considerations, upon noting that $r_1(1) = r_1(2) = r_1(3) = 0$.

For (c), suppose $2 \leqslant x \leqslant 3$, and note that for such $x$,

$$0 \leqslant z(x) \leqslant (x-2) \leqslant 1. \tag{39}$$

We then have

$$\begin{aligned}\frac{2}{8-z(x)}r_2(x) &= 2^{x-2} - 2\left(\frac{4+z(x)}{8-z(x)}\right) = 2^{x-2} - \frac{1+\frac{z(x)}{4}}{1-\frac{z(x)}{8}}\\ &= 2^{x-2} - \left(1+\frac{z(x)}{4}\right)\left(1+\frac{z(x)}{8}\frac{1}{1-\frac{z(x)}{8}}\right)\\ &\geqslant 2^{x-2} - \left(1+\frac{z(x)}{4}\right)\left(1+\frac{z(x)}{7}\right)\\ &\geqslant 2^{x-2} - \left(1+\frac{12}{28}z(x)\right). \qquad (40)\end{aligned}$$

The two inequalities in (40) follow from (39). Expanding about $x = 2$, and employing (39), this gives

$$\begin{aligned}\frac{2}{8-z(x)}r_2(x) &\geqslant 1 + (x-2)\ln 2 - \left(1+\frac{12}{28}(x-2)\right)\\ &= \left(\ln 2 - \frac{12}{28}\right)(x-2) \geqslant 0, \qquad (41)\end{aligned}$$

and part (c) follows. □

We are now in a position to prove Theorem 1.

**Proof of Theorem 1.** Set $h \equiv 0$, and note that for $c \geqslant 0$ (i.e., $k_c \geqslant 1$), $B$ and $g$ defined by $B(x) = x^{k_c}$ and $g(x) = \frac{c}{x^2}$ satisfy

$$B''(x) = k_c(k_c-1)x^{k_c-2} = cx^{k_c-2} = \left(\frac{c}{x^2}\right)x^{k_c} = g(x)B(x), \qquad (42)$$

and hence (7) is satisfied. Now, for $x \geqslant 1$,

$$B^{(4)}(x) = k_c(k_c-1)(k_c-2)(k_c-3)x^{k_c-4}\begin{cases}\leqslant 0, & \text{if } 2 < c < 6,\\ \geqslant 0, & \text{otherwise},\end{cases} \qquad (43)$$

thus $B'''(x)$ is nonincreasing on $x \geqslant 1$, when $2 < c < 6$ (i.e., $2 < k_c < 3$) and nondecreasing when $0 \leqslant c \leqslant 2$ or $c \geqslant 6$. Note that $B(2) = 2^{k_c}$, $B(1) = 1$, $B(0) = 0$, and $g(1) = c$.

**Case 1.** $c \geqslant 6$ or $0 \leqslant c \leqslant 2$ (i.e., $k_c \geqslant 3$ or $1 \leqslant k_c \leqslant 2$). Here, $\min\{B(2) - B(1), B(1) - B(0)\} = 1$ and by Lemma 2(b),

$$\min\left\{\frac{B(2)-B(1)}{1+g(1)}, B(1)-B(0)\right\} = \min\left\{\frac{2^{k_c}-1}{1+c}, 1\right\} = 1. \qquad (44)$$

Applying Corollary 1 with $c_0 = c_1 = 1$ gives (2).

**Case 2.** $2 < c < 6$ (i.e., $2 < k_c < 3$). Here,

$$B'''(x) = k_c(k_c-1)(k_c-2)x^{k_c-3}, \qquad (45)$$

thus $B'''(x) \downarrow 0$ as $x$ tends to infinity. Taking $C = -\frac{1}{6}(B'''(2) + B'''(1))$, we have

$$\min\{B(2) - B(1) + C, B(1) - B(0)\}$$
$$= \min\left\{B(2) - B(1) - \frac{1}{6}\bigl(B'''(2) + B'''(1)\bigr), 1\right\} = \min\{J(c), 1\} = 1, \tag{46}$$

where

$$J(c) \stackrel{\text{def}}{=} 2^{k_c} - 1 - \frac{1}{6}k_c(k_c - 1)(k_c - 2)\bigl(2^{k_c-3} + 1\bigr). \tag{47}$$

The last equality in (46) follows by Lemma 2(c). (In fact $J(c) \geqslant 3$.)

Also, since $J(c) \leqslant 2^{k_c} - 1$, by Lemma 2(a) we have

$$\min\left\{\frac{B(2) - B(1) + C}{1 + g(1)}, B(1) - B(0)\right\} = \min\left\{\frac{J(c)}{1 + c}, 1\right\} = \frac{J(c)}{1 + c}. \tag{48}$$

Applying Corollary 2, with $c_0 = 1$ and $c_1 = J(c)/(1 + c) = \chi_c$, gives (2). □

**Remark.** After completion of this manuscript, Prof. Stević kindly shared with us a preliminary draft of a short note [9] also confirming his conjecture in [8]. The result therein is of a purely asymptotic nature, whereas here we are interested in explicit and applicable bounds. Since the question provided some of our original motivation and the bound in this case is quite simple and informative, we have chosen to leave our handling of his question among our examples. The interested reader is encouraged to seek out [9] for a different perspective on the particular case when $g(i) = c/i^2$ for some $c \geqslant 0$, all $i \geqslant 1$, and $h \equiv 0$ in (1).

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## References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1972.
[2] V.B. Demidovič, Conditions for boundedness of the solutions of second-order linear difference equations with periodic coefficients, Vestnik Moskov. Univ. Ser. I Mat. Meh. 25 (1970) 15–22.
[3] E. Kurpinar, On the boundedness of solutions of second-order difference equation with monotonical coefficients, J. Fac. Sci. Ege Univ. Fen. Fak. Derg. Ser. A–B 18 (1995) 39–42.
[4] E. Kurpinar, G.Sh. Guseinov, The boundedness of solutions of second-order difference equations, Indian J. Math. 37 (2) (1995) 113–122.
[5] D.S. Mitrinović, Analytic Inequalities, in cooperation with P.M. Vasić, Grundlehren Math. Wiss., Band 1965, Springer-Verlag, New York, 1970.
[6] F.W.J. Olver, Error bounds for linear recurrence relations, Math. Comp. 50 (1988) 481–499.
[7] S. Stević, Growth theorems for homogeneous second-order difference equations, ANZIAM J. 43 (2002) 559–566.
[8] S. Stević, Asymptotic behavior of second-order difference equations, ANZIAM J. 46 (2004) 157–170.
[9] S. Stević, Asymptotic behavior of a second-order difference equation, preprint, 2004.
[10] W.F. Trench, Asymptotic behavior of solutions of a linear second-order difference equation, in: Asymptotic Methods in Analysis and Combinatorics, J. Comput. Appl. Math. 41 (1992) 95–103.

[11] W.F. Trench, Linear perturbations of a nonoscillatory second-order difference equation, J. Math. Anal. Appl. 255 (2001) 627–635.

[12] W.F. Trench, Linear perturbations of a nonoscillatory second-order difference equation, II, J. Math. Anal. Appl. 273 (2002) 548–557.

[13] R. Wong, H. Li, Asymptotic expansions for second-order linear difference equations, in: Asymptotic Methods in Analysis and Combinatorics, J. Comput. Appl. Math. 41 (1992) 65–94.